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LETTER TO THE EDITOR

Bohmian mechanics with discrete operators

Ross Hyman[1], Shane A Caldwell[2] and Edward Dalton[2]

[1] Center for Strategic Research, AFL-CIO, 940 W Adams, Suite 403, Chicago, IL 60607, USA
[2] Department of Physics, DePaul University, Chicago, IL 60614-3504, USA

E-mail: rhyman@aflcio.org

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Abstract
A deterministic and time reversible Bohmian mechanics for operators with continuous and discrete spectra is presented. Randomness enters only through initial conditions. Operators with discrete spectra are incorporated into Bohmian mechanics by associating with each operator a continuous variable in which a finite range of the continuous variable corresponds to the same discrete eigenvalue. In this way a deterministic and time reversible Bohmian mechanics can handle the creation and annihilation of particles. The formalism does not depend on the details of the Hamiltonian. Furthermore, many consistent choices are available for the dynamics. Examples are given and generalizations are discussed.

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1. Introduction

In 1951 David Bohm [1] introduced an extension of quantum mechanics, now called de Broglie Bohm theory or Bohmian mechanics, in which particles have definite positions and velocities at all times, including between measurements [2]. He also showed how the same idea can be applied to Bose field configurations. Bohmian mechanics agrees with all of the experimental predictions of the Copenhagen interpretation of quantum mechanics but has the advantages of a smooth transition to classical mechanics, the absence of wavefunction collapse, and not having to separate the universe into quantum systems and classical measuring devices. Furthermore, the dynamics is deterministic and time reversible. Randomness enters only in the initial conditions of the particle and field configurations. Bohm's formulation relies on the fact that the quantum operators for particle positions and Bose field configurations have continuous spectra. In this letter we generalize Bohmian mechanics so that it can handle any set of commuting operators with continuous or discrete spectra. A potential application of this work is a way to accommodate second quantized fermionic field theories without invoking explicit particle trajectories. With this formalism, we can, following a suggestion by Bell [3], divide space into chunks and consider the discrete charge in each chunk, the number of electrons minus the number of positrons in each chunk, as the primary variables. Using the mechanics described in this letter, the dynamics of the charge in each chunk is deterministic, time reversible, and the results of identical experiments agree with quantum mechanical predictions.

In Bohmian mechanics, the quantum state and Hamiltonian do not constitute a complete description of a physical system as they do in the Copenhagen interpretation of quantum mechanics. Bohmian mechanics singles out a set of dynamical variables associated with a particular set of commuting operators (e.g., the positions of all of the particles) that are needed in addition to the quantum state and Hamiltonian to constitute a complete description of the physical system at all times. J S Bell coined the term 'beables', short for 'maybeables' for those operators promoted to this special status [3]. The hesitancy is built into the word to indicate, somewhat analogously to gauge freedom, that the set of commuting operators is not uniquely determined by experiment. Bohm expressed his mechanics in first quantized notation. His formalism required that all of the beables have continuous spectra, which is fine when electron position operators are the beables. However, the Dirac equation presents a problem in this formalism since it requires an infinite number of negative energy electrons in the vacuum, so the most naive extension of Bohmian mechanics to the relativistic realm requires the calculation of an infinite number of trajectories even when there is nothing really there. If instead one thinks of the vacuum as a vacuum, and moves to a second quantized representation, then one is faced with the possibility of electron–positron pair creation and annihilation. This cannot be straightforwardly treated with continuous variables alone since the number of particles is not a continuous variable.

Since Bohm's 1951 work several varieties of Bohmian dynamics for relativistic quantum field theory and several varieties of dynamics for beables with discrete spectra have been proposed [2–9]. Durr *et al* [2] retain Bohmian particle trajectories and posit an infinite number of spaces, each with a definite number of particles of each species (i.e. a definite number of electrons and a definite number of positrons). In each space the particles have definite positions and velocities determined by Bohmian mechanics. Additional discrete beables, one for the total number of particles of each species, are added to the system. The discrete beables hop around stochastically (using stochastic dynamics developed by Bell [3] for a different set of beables) so that at each instant of time the system has a definite number of electrons and a definite number of positrons, and the positions of these particles are determined by the positions of the particles for that particular fixed-particle-number space. An alternative formalism by Nikolić [5, 6] removes the stochastic element from the dynamics between measurements and only assigns a definite number of particles to the system when a measurement is made. All trajectory based forms of Bohmian mechanics, for bosons or fermions, assume that the indistinguishability of identical particles does not extend to the Bohmian level. We will not pursue these, or the other formalisms of trajectory based Bohmian mechanics for fermions in our letter.

An alternative stochastic formalism for dealing with relativist fermion field theory was developed by Bell in 1984 [3]. Bell developed a dynamics for beables derived from any set of commuting operators with discrete spectra. He applied it to fermion field theory by forsaking particle trajectories altogether and instead divided space into chunks and considered the charge in each chunk, which is the number of electrons minus the number of positrons in each chunk, as the beables. Bell's mechanics is stochastic. He was dissatisfied with this since '... the reversibility of the Schrödinger equation strongly suggests that quantum mechanics is not fundamentally stochastic in nature. However I suspect that the stochastic element introduced here goes away in some sense in the continuum limit' [3]. Bell's suspicion has been explored by others [4] and has been found to be essentially correct. Bell's stochastic dynamics agrees with Bohm's deterministic dynamics for fixed particle systems as the volume of each chunk is taken to zero. Other authors have proposed deterministic dynamics for fermionic quantum field theory by finding some way to parametrize the discrete charges with continuous variables [8, 9]. However, the construction of a deterministic time reversible dynamics for any Hamiltonian and for any set of commuting beables with continuous and discrete spectra has not, to our knowledge, been addressed before (however, see [6] for a proposal similar to ours for beables with continuous spectra).

In this letter we introduce a non-stochastic, deterministic, and time-reversible Bohmian mechanics for any Hamiltonian and for any set of commuting beable operators with discrete and/or continuous spectra. In principle, the formalism can be applied to Bell's choice of chunk charge beables. The determinism and time reversible invariance is present at the course grained level. No continuous limit is necessary (i.e. the volume of the chunks need not be taken to zero), no limitation on the total number of particles is required, and no modification of the Hamiltonian need be made. We have attempted to describe the formalism in as general a way as possible, pointing out at each step where the quantum mechanics undetermines the formalism so that many consistent choices are available leading to different beable dynamics. The Bohmian interpretation of quantum mechanics proposes that it is possible for something to exist that is beyond our capability through experiment to completely discern (i.e. the correct beable dynamics out of all of the infinite possibilities of beable dynamics that are consistent with experiment). The Copenhagen interpretation, on the other hand, proclaims that if it is beyond our capability through experiment to completely discern something, then it simply does not exist (i.e. the dynamics of anything in between measurements). Perhaps it is not a coincidence that many of the relativistic versions of Bohmian mechanics require an interpretation of special relativity in which an absolute frame of reference, although experimentally undetectable, actually exists, in contrast to the conventional interpretation, in which an absolute frame of reference, since it is experimentally undetectable is held not to exist [10].

In section 2 of this letter we present a generalization of Bohmian mechanics for any set of commuting operators with continuous spectra. The formalism is amenable to operators with discrete spectra, and in section 3 we incorporate operators with discrete spectra into the formalism. In section 4 we present the exactly solvable case of Bohmian mechanics for one beable, and an intriguing visualization of Bohmian mechanics for any number of beables. Section 5 contains a summary of our results.

2. Bohmian mechanics with projection operators

In this section we present Bohmian mechanics in a generalized way, making extensive use of projection operators, which will allow us to incorporate operators with discrete spectra. The generalization agrees with Bohm's original formulation when the beables are particle position operators but also allows for any choice of commuting operators $\hat{\xi}_\ell$, $\ell = 1, 2, \ldots, L$, $[\hat{\xi}_\ell, \hat{\xi}_{\ell'}] = 0$ for the beables, provided that the choice is sufficient to describe the status of all possible measurement devices. The operators can have continuous or discrete spectra. In this section we will deal only with operators with continuous spectra and extend the formalism to discrete operators in the following section.

If $\hat{\xi}_\ell$ has continuous spectra we can express it as

$$\hat{\xi}_\ell = \int \mathrm{d}\lambda_\ell \, \xi_\ell(\lambda_\ell) \hat{P}_\ell(\lambda_\ell) \tag{2.1}$$

where λ_ℓ parametrizes the eigenstates of $\hat{\xi}_\ell$, the integral is taken over the entire range of λ_ℓ, and $\xi_\ell(\lambda_\ell)$ is the eigenvalue of $\hat{\xi}_\ell$ associated with the eigenstates labelled by λ_ℓ.

$$\hat{\xi}_\ell|\lambda_\ell, q, \ell\rangle = \xi_\ell(\lambda_\ell)|\lambda_\ell, q, \ell\rangle \tag{2.2}$$

where q distinguishes states with the same eigenvalue of $\hat{\xi}_\ell$. The projection operator for the eigenstates associated with $\xi_\ell(\lambda_\ell)$ is

$$\hat{P}_\ell(\lambda_\ell) = \sum_q |\lambda_\ell, q, \ell\rangle\langle\lambda_\ell, q, \ell|. \tag{2.3}$$

In this expression the sum over q represents the sum or integral over states with the same eigenvalue of $\hat{\xi}_\ell$. The simplest $\xi_\ell(\lambda_\ell)$ function for an operator with continuous spectra is $\xi_\ell(\lambda_\ell) = \lambda_\ell$ in which case λ_ℓ has units of ξ_ℓ. For this case the projection operator density, $\hat{P}_\ell(\lambda_\ell)$, takes on the particularly simple form

$$\hat{P}_\ell(\lambda_\ell) = \delta(\lambda_\ell - \hat{\xi}_\ell). \tag{2.4}$$

For operators with discrete spectra, which we cover in the following section, we will find it convenient to associate a range of λ variables with a single ξ eigenvalue.

Bohmian mechanics describes the dynamics of a set of $\lambda_\ell(t)$ (that is, $\lambda_1, \lambda_2, \lambda_3, \ldots$, which we denote collectively by Λ) and thereby a set of $\xi_\ell(\lambda_\ell(t))$ (which we denote collectively by Ξ) which represents the physical values of these values at time t, regardless if a measurement is made or not. The quantum probability distribution of a particular Λ configuration is conveniently written in terms of the projection operators

$$P(\Lambda, t) = \langle t|\prod_{\ell=1}^{L}\hat{P}_\ell(\lambda_\ell)|t\rangle. \tag{2.5}$$

This is a probability distribution since

$$\int \mathrm{d}\lambda_1 \int \mathrm{d}\lambda_2 \cdots \int \mathrm{d}\lambda_L\, f(\Xi)P(\Lambda, t) = \langle t|f(\hat{\Xi})|t\rangle. \tag{2.6}$$

The result is unambiguous since we require that the $\hat{\xi}$ all commute with each other. The probability distribution has all the properties required of a classical probability distribution. The integral of the probability distribution taken over all of Λ space is 1, and the probability distribution is real and non-negative provided that all of the projectors in the operator product commute with each other. The projectors will commute if the $\hat{\xi}_\ell$ all commute with each other, which is why we made this requirement for our set of beables.

The quantum state is in general not an eigenstate of the $\hat{\Xi}$. It is propagated forward in time as in conventional quantum mechanics

$$\mathrm{i}\hbar\frac{\mathrm{d}|t\rangle}{\mathrm{d}t} = \hat{H}|t\rangle. \tag{2.7}$$

Additionally, the Λ configuration is propagated forward in time with the first-order equations

$$\frac{\mathrm{d}\lambda_\ell(t)}{\mathrm{d}t} = v_\ell(\Lambda, t), \tag{2.8}$$

where the $v_\ell(\Lambda, t)$ are chosen so that the classical probability distribution of the Λ configuration of an ensemble of identical experiments

$$P_c(\Lambda, t) = \int \mathrm{d}\lambda_1'(0) \int \mathrm{d}\lambda_2'(0) \cdots \int \mathrm{d}\lambda_L'(0) P_c(\Lambda', 0) \prod_{\ell=1}^{L} \delta(\lambda_\ell - \lambda_\ell'(t, \{\Lambda'(0)\}) \tag{2.9}$$

agrees with the quantum probability distribution at all time provided they are in agreement at any one time. If the beables are chosen such that all measurements are the measurements of the beables, this guarantees that the results of Bohmian mechanics are consistent with the results of conventional quantum mechanics.

Using Bohmian mechanics equation (2.8), the time dynamics of the classical probability distribution is

$$\frac{\mathrm{d}P_c(\Lambda,t)}{\mathrm{d}t} = -\sum_{\ell=1}^{L} \frac{\partial P_c(\Lambda,t)v_\ell(\Lambda,t)}{\partial\lambda_\ell}. \tag{2.10}$$

Whereas the time derivative of the quantum probability distribution on the other hand is

$$\frac{\partial P(\Lambda,t)}{\partial t} = \sum_{\ell=1}^{L}\langle t|\left(\prod_{j=1}^{\ell-1}\hat{P}_j(\lambda_j)\right)\frac{1}{\mathrm{i}\hbar}[\hat{P}_\ell(\lambda_\ell),\hat{H}]\left(\prod_{k=\ell+1}^{L}\hat{P}_k(\lambda_k)\right)|t\rangle. \tag{2.11}$$

Following David Bohm's insight, we note that if the classical and quantum probability distributions agree at any particular time then they agree for all time provided that the $v_\ell(\{\lambda\})$ are chosen so that the two time derivatives equations (2.10) and (2.11) are equal. This is what we do now.

Our goal is to rewrite equation (2.11) in the form

$$\frac{\partial P(\Lambda,t)}{\partial t} = -\sum_{\ell=1}^{L}\frac{\partial J_\ell(\Lambda,t)}{\partial\lambda_\ell}, \tag{2.12}$$

with $J_\ell(\Lambda,t)$ real. If this can be accomplished then we can set

$$v_\ell(\Lambda) = \frac{J_\ell(\Lambda,t)}{P(\Lambda,t)} \tag{2.13}$$

and we will have determined a consistent Bohmian dynamics. Associating the ℓ terms in both expressions we have

$$\frac{\partial J_\ell(\Lambda,t)}{\partial\lambda_\ell} = -\langle t|\left(\prod_{j=1}^{\ell-1}\hat{P}_j(\lambda_j)\right)\frac{1}{\mathrm{i}\hbar}[\hat{P}_\ell(\lambda_\ell),\hat{H}]\left(\prod_{k=\ell+1}^{L}\hat{P}_k(\lambda_k)\right)|t\rangle. \tag{2.14}$$

Ignoring for the moment the possibility that the right-hand side of equation (2.14) is not real, we write

$$J_\ell(\Lambda,t) = \langle t|\left(\prod_{\ell'=1}^{\ell-1}\hat{P}_{\ell'}(\lambda_{\ell'})\right)\hat{J}_\ell(\lambda_\ell)\left(\prod_{\ell''=\ell+1}^{L}\hat{P}_{\ell''}(\lambda_{\ell''})\right)|t\rangle, \tag{2.15}$$

in which

$$\frac{\mathrm{d}\hat{J}_\ell(\lambda)}{\mathrm{d}\lambda} = -\frac{1}{\mathrm{i}\hbar}[\hat{P}_\ell(\lambda),\hat{H}] \tag{2.16}$$

which is easily integrated to

$$\hat{J}_\ell(\lambda_\ell) = \frac{1}{\mathrm{i}\hbar}[\hat{G}_\ell(\lambda_\ell),\hat{H}] = -\frac{1}{\mathrm{i}\hbar}[\hat{L}_\ell(\lambda_\ell),\hat{H}], \tag{2.17}$$

where

$$\hat{G}_\ell(\lambda_\ell) = \int_{\lambda_\ell}\mathrm{d}\lambda'\,\hat{P}_\ell(\lambda') \tag{2.18}$$

and

$$\hat{L}_\ell(\lambda_\ell) = \int^{\lambda_\ell}\mathrm{d}\lambda'\,\hat{P}_\ell(\lambda') \tag{2.19}$$

are projection operators for all states greater than or less than λ_ℓ respectively. The current operator, $\hat{J}_\ell$, can be generalized to periodic beables such as the position of a bead on a ring with

$$\hat{J}_\ell(\lambda_\ell) = \frac{1}{\mathrm{i}\hbar} \int \mathrm{d}\lambda' \int \mathrm{d}\lambda'' \hat{P}_\ell(\lambda'_\ell) \hat{H} \hat{P}_\ell(\lambda''_\ell) f(\lambda'_\ell, \lambda_\ell, \lambda''_\ell) \tag{2.20}$$

where $f(\lambda'_\ell, \lambda_\ell, \lambda''_\ell) = +1(-1)$ if there is a non-crossing path that goes from λ'_ℓ to λ''_ℓ through λ_ℓ in the positive (negative) direction and $f(\lambda'_\ell, \lambda_\ell, \lambda''_\ell) = 0$ if there is no such path. We will not use this generalization in this letter.

The right-hand side of equation (2.14) is not guaranteed to be real unless $[[\hat{P}_\ell, \hat{H}], \hat{P}_{\ell'}] = 0$ for $\ell \neq \ell'$ which is true if $[[\hat{\xi}_\ell, \hat{H}], \hat{\xi}_{\ell'}] = 0$ for $\ell \neq \ell'$. If this is not the case we can make it real simply by taking the real part

$$J_\ell(\Lambda, t) = \mathrm{Re}\left(\langle t| \left(\prod_{\ell'=1}^{\ell-1} \hat{P}_{\ell'}(\lambda_{\ell'}) \right) \hat{J}_\ell(\lambda_\ell) \left(\prod_{\ell''=\ell+1}^{L} \hat{P}_{\ell''}(\lambda_{\ell''}) \right) |t\rangle \right), \tag{2.21}$$

but this picks out a particular order of the operators for special treatment. A more democratic way to guarantee that the current is real is the symmetric average

$$J_\ell(\Lambda, t) = \langle t| S \left\{ \left(\prod_{\ell'=1}^{\ell-1} \hat{P}_{\ell'}(\lambda_{\ell'}) \right) \hat{J}_\ell(\lambda_\ell) \left(\prod_{\ell''=\ell+1}^{L} \hat{P}_{\ell''}(\lambda_{\ell''}) \right) \right\} |t\rangle \tag{2.22}$$

where $S\{\cdots\}$ implies a symmetric average of all of the operators inside the braces. For example if $L = 3$,

$$J_1(\lambda_1, \lambda_2, \lambda_3) = \tfrac{1}{6} \langle t| 2\hat{J}_1 \hat{P}_2 \hat{P}_3 + \hat{P}_2 \hat{J}_1 \hat{P}_3 + \hat{P}_3 \hat{J}_1 \hat{P}_2 + 2\hat{P}_2 \hat{P}_3 \hat{J}_1 |t\rangle, \tag{2.23}$$

where we have used the fact that the P commute to combine some terms. This freedom in the choice of J_ℓ is a particular example of a more general freedom. We can add any function Q_ℓ to the probability current density, J_ℓ, such that $\sum_\ell \mathrm{d}Q_\ell/\mathrm{d}\lambda_\ell = 0$. For an alternative choice of J_ℓ covered within the scope of this freedom see [6]. This is the second way that the dynamics are not unique (the first being the choice of which operators to anoint to beable status).

Our final expression for the λ dynamics is

$$\frac{\mathrm{d}\lambda_\ell(t)}{\mathrm{d}t} = v_\ell(\Lambda, t) = \frac{\langle t| S\{ \left(\prod_{\ell'=1}^{\ell-1} \hat{P}_{\ell'}(\lambda_{\ell'}) \right) \hat{J}_\ell(\lambda_\ell) \left(\prod_{\ell''=\ell+1}^{L} \hat{P}_{\ell''}(\lambda_{\ell''}) \right) \} |t\rangle}{\langle t| \prod_{\ell=1}^{L} \hat{P}_\ell(\lambda_\ell) |t\rangle}. \tag{2.24}$$

An equivalent way to write the equations of Bohmian mechanics is to use the Heisenberg representation in which the quantum state does not change with time but any operator $\hat{A}(t)$ depends on time via

$$\hat{A}(t) = \mathrm{e}^{-\hat{H}t/\mathrm{i}\hbar} \hat{A}\, \mathrm{e}^{\hat{H}t/\mathrm{i}\hbar}. \tag{2.25}$$

The equations of Bohmian mechanics in the Heisenberg representation are

$$\langle 0| S \left\{ \left(\prod_{\ell'=1}^{\ell-1} \hat{P}_{\ell'}(\lambda_{\ell'}, t) \right) \mathrm{d}\hat{L}_\ell(\lambda_\ell, t) \left(\prod_{\ell''=\ell+1}^{L} \hat{P}_{\ell''}(\lambda_{\ell''}, t) \right) \right\} |0\rangle = 0, \tag{2.26}$$

where $\hat{L}_\ell(\lambda_\ell, t)$ is defined by equation (2.19) and

$$\mathrm{d}\hat{L}_\ell(\lambda_\ell, t) = \mathrm{d}t \frac{\partial \hat{L}_\ell(\lambda_\ell, t)}{\partial t} + \mathrm{d}\lambda_\ell \frac{\partial \hat{L}_\ell(\lambda_\ell, t)}{\partial \lambda_\ell}, \tag{2.27}$$

in which

$$\frac{\partial \hat{L}_\ell(\lambda_\ell, t)}{\partial t} = \frac{1}{\mathrm{i}\hbar} [\hat{L}_\ell(\lambda_\ell, t), \hat{H}] = -\hat{J}_\ell(\lambda_\ell, t), \tag{2.28}$$

and

$$\frac{\partial \hat{L}_\ell(\lambda_\ell, t)}{\partial \lambda_\ell} = \hat{P}_\ell(\lambda_\ell, t). \tag{2.29}$$

In the Schrödinger representation, since the quantum state changes with time, one is tempted to think of it as a dynamic variable just like the beables and wonder why the beables depend on the quantum state but not the other way around. In the Heisenberg representation the quantum state is not a dynamic variable so this apparent asymmetry does not arise.

If an ensemble of identical experiments are performed, in which the Λ configurations at a particular time for each experiment are taken at random from the quantum distribution equation (2.5) then if the Λ configurations are propagated forwards and backwards in time via equation (2.24) or equation (2.26) then for all other times the probability distribution of the Λ configurations over the ensemble equation (2.9) will be equal to the quantum distribution. Provided that the set of anointed operators is sufficient to describe the status of all measurement devices, Bohmian mechanics will agree with all results of conventional quantum mechanics without resorting to wavefunction collapse or some other alternative to the Schrödinger dynamics to describe the measurement process as is done in the orthodox Copenhagen interpretation. Bohmian mechanics replaces this with the mystery of how to explain why the classical and quantum probability distributions should agree at any time at all [5].

If all of the λ_ℓ correspond to position coordinates in nonrelativistic quantum mechanics then the conventional form of Bohmian mechanics is recovered from equation (2.24) or equation (2.26). For then the current operators are

$$\hat{J}_\ell(\lambda_\ell) = \frac{1}{2}\left[\frac{\hat{p}_\ell}{m_\ell}, \delta(\lambda_\ell - \hat{x}_\ell)\right]_+ . \tag{2.30}$$

This is the current operator in conventional Bohmian mechanics so the equivalence with the traditional formalism is proved. The present formalism is more flexible than the traditional formalism and can be easily generalized to account for beable operators with discrete spectra, which is what we consider in the following section.

3. Fitting the discrete square peg into the continuous round hole

In this section we incorporate beables derived from operators with discrete spectra into Bohmian mechanics. Doing so, we are immediately faced with the question of how to retain determinism which Bell sacrificed with regret. The problem is most apparent when the initial quantum state is an eigenstate of each ξ_ℓ for then the initial Ξ configuration is uniquely determined so there appears to be no room for any randomness in the initial conditions. The quantum time dynamics will immediately make the quantum state a superposition of Ξ states so the Bohmian probability distribution for the beables must develop some spreading to agree with its quantum counterpart. Since there is no randomness in the initial conditions, it appears that randomness must enter through the dynamics and therefore determinism must be sacrificed. Strictly speaking the same problem can arise for operators with continuous spectra. But for continuous spectra one can wiggle one's way out by asserting that in any actual system the quantum state is never exactly in an eigenstate of all of the operators. There is always some spreading for whatever reason. For example Bohmian mechanics with particle position beables is in trouble when the wavefunction is initially a delta function, the eigenstate of the position operator. But Bohmain mechanics is fine for even the tiniest amount of wavefunction spreading over a continuous range of particle positions. As Nikolić [12] has noted, a Dirac delta function is not a proper wavefunction since its square is not normalizable.

The proper way to handle the eigenstate of an operator with continuous spectra in conventional quantum mechanics is to use a normalizable wavefunction with some tiny amount of spreading and take the limit of zero spreading only at the end of a calculation. Since this spreading is needed in conventional quantum mechanics, we should not be surprised that it is also needed for Bohmian mechanics. However, eigenstates of operators with discrete spectra are normalizable, so conventional quantum mechanics does not require any spreading and we feel obliged not to require it for Bohmian mechanics either.

A way to retain deterministic dynamics with discrete operators, for arbitrary normalizable initial quantum state is to assign finite ranges of λ_ℓ to the same eigenvalue of $\hat{\xi}_\ell$. ξ_ℓ sits still when λ_ℓ is moving smoothly through a region that corresponds to the same eigenvalue and ξ_ℓ makes a sudden hop to a new eigenvalue when λ_ℓ smoothly moves from one eigenvalue region to another. The problem of determinism is solved by this devise since there are now many Λ configurations corresponding to the same Ξ configuration. Randomness in the initial distribution of Λ is sufficient to account for the randomness of all subsequent configurations propagated forward with deterministic dynamics. It might appear that for operators with discrete spectra, λ_ℓ can legitimately be called a hidden variable since the particular value of λ_ℓ inside an eigenvalue range is unobservable and therefore 'hidden'. But λ_ℓ is observable at those times when the ξ_ℓ hops to a new value since there is a unique value of λ_ℓ for each transition. Therefore λ_ℓ encodes observable information about previous transitions and the times that they occurred. We now construct the explicit Bohmian mechanics for the discrete case.

If $\hat{\xi}_\ell$ has discrete spectra we can express it as

$$\hat{\xi}_\ell = \sum_n \xi_{\ell n} \hat{P}_\ell(n), \tag{3.1}$$

where $\xi_{\ell n}$ is the nth eigenvalue of $\hat{\xi}_\ell$ and the projection operator is

$$\hat{P}_\ell(n) = \sum_q |n, q, \ell\rangle\langle n, q, \ell|, \tag{3.2}$$

where as in the continuous case, the sum over q represents the sum or integral over states with the same eigenvalue of $\hat{\xi}_\ell$.

We now seek to express equation (3.1) in the continuous form (2.1) and define an appropriate $\xi_\ell(\lambda_\ell)$ function and projection operator $\hat{P}_\ell(\lambda_\ell)$ so that we may carry over equation (2.24) or equation (2.26) unchanged for the dynamics. There are many ways to do this. Technically, $\xi_\ell(\lambda_\ell) = \lambda_\ell$ and $\hat{P}_\ell(\lambda) = \delta(\lambda - \hat{\xi}_\ell)$ as in the continuous case does the job. But this leads to zero probability for λ_ℓ not equal to an eigenvalue. We can correct for this by smearing out the delta function over a range of λ_ℓ so that $\hat{P}_\ell(\lambda_\ell)$ is not zero between eigenvalues and a range of λ_ℓ corresponds to the same state. There are innumerable ways to parametrize ξ_ℓ to achieve this. Here is one way. Define the ξ_ℓ function

$$\xi_\ell(\lambda_\ell) = \xi_{\ell,n} \qquad n = n(\lambda_\ell), \tag{3.3}$$

where λ_ℓ has no units and $n(\lambda_\ell)$ is the closest integer to λ_ℓ. With this $\xi_\ell(\lambda_\ell)$ function we achieve agreement between equations (3.1) and (2.1) using

$$\hat{P}_\ell(\lambda_\ell) = \hat{P}_\ell(n(\lambda_\ell)), \tag{3.4}$$

where $\hat{P}_\ell(n(\lambda_\ell)) = 0$ if there is no eigenvalue associated with the integer $n(\lambda_\ell)$. For discrete spectra with integer eigenvalues we can use the explicit forms

$$\begin{aligned}\hat{G}_\ell(\lambda_\ell) &= \left(n(\lambda_\ell) + \tfrac{1}{2} - \lambda_\ell\right) \hat{P}_\ell(n(\lambda_\ell)) + \sum_{j=n(\lambda_\ell)+1}^{\infty} \hat{P}_\ell(j)\\ \hat{L}_\ell(\lambda_\ell) &= \left(\lambda_\ell - n(\lambda_\ell) + \tfrac{1}{2}\right) \hat{P}_\ell(n(\lambda_\ell)) + \sum_{j=-\infty}^{n(\lambda_\ell)-1} \hat{P}_\ell(j),\end{aligned} \tag{3.5}$$

in the expressions for the current operator $\hat{J}_\ell$.

Using these definitions in the velocity expression (2.24) or equation (2.26) and keeping in mind that the physical values of ξ_ℓ with discrete spectra are determined by equation (3.3), we have defined a deterministic and time reversible Bohmian mechanics for operators with discrete and continuous spectra. The initial Λ configuration is taken from the quantum probability distribution (2.5) with equation (3.4) used for projectors for discrete operators. This means that the particular λ_ℓ for a given $\xi_{\ell,n}$ is chosen at random from a distribution spread uniformly from $n - 1/2$ to $n + 1/2$. We are free to order the eigenvalues along the λ_ℓ line any way we like, a freedom that also exists in the continuous case. This is the third way that the dynamics is not unique (although this is really a variation of the first way the dynamics is not unique, the choice of beables, since reordering the eigenstates of an operator is no different than selecting another operator with the same eigenstates but different eigenvalues). The choice of ordering of the eigenvalues in Λ space for each beable operator profoundly effects the dynamics since, for example a system in eigenvalue state 2 can only get to eigenvalue state 4 by first passing through eigenvalue state 3. For some systems the Hamiltonian suggests a particular order, but perhaps there are Hamiltonians with transition elements between eigenvalues that are far apart in Λ space, for any ordering that one chooses. This is a possible disadvantage of this method, which is not shared by the stochastic Bell scheme.

In the following section we show how the formalism of this letter works for some simple examples. We also present a visualization of Bohmian mechanics in which the dynamics occurs not in Λ space, but in what we call bubble space. Bohmian mechanics in bubble space is perhaps even more undetermined than Bohmian mechanics in Λ space.

4. Bohmian mechanics in bubble space

Bohmian mechanics is integrable for the case in which there is only one operator promoted to beable status. For in that case

$$\langle 0|\,\mathrm{d}\hat{L}(\lambda, t)|0\rangle = \mathrm{d}\langle 0|\hat{L}(\lambda, t)|0\rangle = 0, \tag{4.1}$$

so the solution is

$$\langle 0|\hat{L}(\lambda, t)|0\rangle = \langle t|\hat{L}(\lambda)|t\rangle = L_0. \tag{4.2}$$

This equation has the following visual interpretation for the case of a beable with eigenvalues a, b, c, etc. The integration constant, L_0, is uniformly distributed on a line between zero and one and sits immovable on the line. Associate the portion of the line from zero to $\langle t|\hat{P}_a|t\rangle$ with beable value a, the portion of the line from $\langle t|\hat{P}_a|t\rangle$ to $\langle t|\hat{P}_a|t\rangle + \langle t|\hat{P}_b|t\rangle$ with beable value b, the portion of the line from $\langle t|\hat{P}_a|t\rangle + \langle t|\hat{P}_b|t\rangle$ to $\langle t|\hat{P}_a|t\rangle + \langle t|\hat{P}_b|t\rangle + \langle t|\hat{P}_c|t\rangle$ with beable value c, etc. These boundaries change with time. The value of the beable at any time is the value associated with the portion of the line that L_0 sits on at that time. As an example, consider a two-state case in which an operator $\hat{\xi}$, with eigenvalues ± 1 and projectors $\hat{P}_\pm$, is the beable (we have chosen a slightly different parametrization of the eigenvalues than we did in the previous section to take advantage of the symmetry in the two-state case). The Bohmian dynamics dictate that $\xi = -1$ for $\langle t|\hat{P}_-|t\rangle > L_0$ and $\xi = +1$ for $L_0 > \langle t|\hat{P}_-|t\rangle$. These results can be combined into the equation of motion

$$\xi(t) = \mathrm{sign}(L_0 - \langle t|\hat{P}_-|t\rangle) \tag{4.3}$$

or using $\langle t|\hat{\xi}|t\rangle = 1 - 2\langle t|\hat{P}_-|t\rangle$

$$\xi(t) = \mathrm{sign}(\langle t|\hat{\xi}|t\rangle - \xi_0), \tag{4.4}$$

where $\xi_0 = 1 - 2L_0$ which is uniformly distributed from -1 to 1. Note that ξ_0 encodes observable information about the times t_j that ξ changes its state via $\langle t_j|\hat{\xi}|t_j\rangle = \xi_0$ so it is not 'hidden', although it is uncontrollable. Also, the average value of $\xi(t)$ over all ξ_0 agrees with the quantum expectation value

$$\int_{-1}^{+1} \xi(t) P(\xi_0)\,\mathrm{d}\xi_0 = \langle t|\hat{\xi}|t\rangle \tag{4.5}$$

as is required for Bohmian mechanics to be consistent with quantum mechanics.

The exact solution for one beable suggests a visual interpretation of Bohmian mechanics for any number of beables that allows us to dispense with Λ as we were able to do for the one beable case. For n beables consider an n-dimensional space of n-volume equal to unity with fixed boundaries. The space is divided into several n-dimensional bubbles. Each bubble corresponds to a particular Ξ configuration and the volume of each bubble is the quantum probability of that configuration. Since the probabilities change with time, the bubbles are continuously contracting and expanding against each other. An immovable point is chosen at random in the n-dimensional space. The physical Ξ configuration at time t is the Ξ configuration corresponding to the bubble enclosing the immovable point at time t.

Any dynamics for the bubble interfaces that insures that the bubble volumes correspond to the correct probabilities for all times is a dynamics that will agree with the results of quantum mechanics. Different dynamics assign different ways for how the bubbles expand and contract against each other. One possible bubble interface dynamics is defined by adjusting the shapes of the bubbles so that the bubble volumes correspond to the correct probabilities and that the $(n-1)$-dimensional bubble interface areas are minimized. An exploration of this dynamics and how it relates to the Bohmian mechanics described in this letter is not attempted here. Below we sketch a bubbles dynamics that corresponds to the Bohmian mechanics of this letter and demonstrate it with some simple examples.

Assume that at time t the bubble volumes are correct and that they are shaped and arranged so that an interface between two bubbles exists for configurations of the two bubbles that differ by only one beable and the two values of that one different beable are adjacent values. The interfaces then need to be adjusted with time, in a deterministic time reversible manner, to correspond to the currents defined in this letter. For example, for a three-beable system in which we divide space into three chunks labelled A, B and C, and the beables are the number of electrons minus the number of positrons in each chunk, then two of the bubbles that must share an interface are the bubbles for configurations $(7, 2, -8)$ and $(7, 3, -8)$. At time t the volumes of bubbles $(7, 2, -8)$ and $(7, 3, -8)$ are $\langle t|\hat{P}_\mathrm{A}(7), \hat{P}_\mathrm{B}(2), \hat{P}_\mathrm{C}(-8)|t\rangle$ and $\langle t|\hat{P}_\mathrm{A}(7), \hat{P}_\mathrm{B}(3), \hat{P}_\mathrm{C}(-8)|t\rangle$ respectively. At the time $t + \mathrm{d}t$ the interface between bubbles $(7, 2, -8)$ and $(7, 3, -8)$ must adjust so that the volume of bubble $(7, 3, -8)$ changes by $\mathrm{d}t\, J_\mathrm{B}(7, 2.5, -8)$ and the volume of bubble $(7, 2, -8)$ changes by $-\mathrm{d}t\, J_\mathrm{B}(7, 2.5, -8)$, where

$$J_\mathrm{B}(7, 2.5, -8) = \langle t|\hat{S}\{\hat{P}_\mathrm{A}(7), \hat{J}_\mathrm{A}(2.5), \hat{P}_\mathrm{C}(-8)\}|t\rangle, \tag{4.6}$$

and

$$\hat{J}_{\mathrm{B}}(2.5) = -\frac{1}{\mathrm{i}\hbar}\left[\sum_{j=-\infty}^{2} \hat{P}_{\mathrm{B}}(j), \hat{H}\right], \tag{4.7}$$

in agreement with the current defined previously in equation (3.5). Note that the bubble interfaces move in the direction opposite to the probability current. The new bubble volumes are guaranteed to agree with the quantum probabilities since

$$\begin{aligned}\frac{\mathrm{d}P(7,3,-8)}{\mathrm{d}t} &= J_{\mathrm{A}}(6.5,3,-8) - J_{\mathrm{A}}(7.5,3,-8) + J_{\mathrm{B}}(7,2.5,-8)\\ &\quad - J_{\mathrm{B}}(7,3.5,-8) + J_{\mathrm{C}}(7,3,-8.5) - J_{\mathrm{C}}(7,3,-7.5).\end{aligned} \tag{4.8}$$

Aside from the fractions, such as 2.5, which are merely notational indications that we are concerned about flow between neighbouring integers, no λ values appear except for those corresponding to ξ eigenvalues.

Explicit deterministic and time reversible constructions of bubble interface dynamics for even simple cases, such as the general two-beable system, are probably easier said than done because of the potentially quite complicated structure of bubble interfaces (although, an interface area minimization rule might be a good way to handle the fact that quantum mechanics underdetermines the bubble interface dynamics). However, for certain situations, the bubble interfaces have simple structure and the dynamics is trivial. The general case for one beable, where the interfaces are just points, is described at the beginning of this section. Here is an explicit formula for bubble configuration dynamics for the case of two beables A and B in which each beable can take only two values: 0 and 1. Imagine a circle with area 1 composed of four pie slices labelled counterclockwise along the pie: (0, 0), (1, 0), (1, 1) and (0, 1) in which each pie slice has the appropriate area corresponding to its quantum probability. The four interfaces are straight radii and can be represented by angles. The dynamics of the interface angles are

$$\begin{aligned}\frac{\mathrm{d}\theta(0.5,0)}{\mathrm{d}t} &= -2\pi J_{\mathrm{A}}(0.5,0) \qquad & \frac{\mathrm{d}\theta(1,0.5)}{\mathrm{d}t} &= -2\pi J_{\mathrm{B}}(1,0.5)\\ \frac{\mathrm{d}\theta(0.5,1)}{\mathrm{d}t} &= +2\pi J_{\mathrm{A}}(0.5,1) \qquad & \frac{\mathrm{d}\theta(0,0.5)}{\mathrm{d}t} &= +2\pi J_{\mathrm{B}}(0,0.5).\end{aligned} \tag{4.9}$$

A fixed point is chosen in the pie and the configuration of the system at any time is the configuration of the pie slice that the fixed point sits in at time t. This two-dimensional system is really a four-state periodic one-dimensional system so the interfaces have no structure. Explicit deterministic and time reversible interface dynamics for more general cases, where, for the n-beable system, the interfaces are $(n-1)$-dimensional membranes are beyond the scope of this letter.

5. Summary

In this letter we have generalized Bohmian mechanics so that it can be applied to any Hamiltonian and can incorporate any set of commuting beable operators with discrete and/or continuous spectra. These generalizations are consistent with experiment provided that the choice of beables is sufficient to describe the status of all possible measurement devices. The equations are deterministic and time reversible and agree with Bohm's original formulation for the case of continuous position operators for the beables. In principle, the formalism can be applied to Bell's choice of chunk charge beables. For a given set of beables, the dynamics is not unique since the beable velocities are underdetermined. The solutions for some simple cases suggest an intriguing visualization of Bohmian mechanics for any number of beables in which bubbles representing the beable configurations have volumes equal to the probabilities of the configurations and expand and contract against each other, in an underdetermined way, as the probabilities change with time. An immoveable point is chosen at random in the bubble space and the physical beable configuration at any time is the beable configuration corresponding to the bubble enclosing the immovable point at that time.

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